Accelerated growth in outgoing links in evolving networks: Deterministic versus stochastic picture

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In several real-world networks such as the Internet, World Wide Web, etc., the number of links grow in time in a nonlinear fashion. We consider growing networks in which the number of outgoing links is a nonlinear function of time but new links between older nodes are forbidden. The attachments are made using a preferential attachment scheme. In the deterministic picture, the number of outgoing links $m(t)$ at any time t is taken as $N(t)^{\theta}$ where $N(t)$ is the number of nodes present at that time. The continuum theory predicts a power-law decay of the degree distribution: $P(k) \propto k^{-1-2/(1-\theta)}$, while the degree of the node introduced at time t_i is given by $k(t_i,t)=t_i^{\theta}[t/t_i]^{(1+\theta)/2}$ when the network is evolved till time t. Numerical results show a growth in the degree distribution for small k values at any nonzero θ. In the stochastic picture, $m(t)$ is a random variable. As long as $\langle m(t) \rangle$ is independent of time, the network shows a behavior similar to the Barabási-Albert (BA) model. Different results are obtained when $\langle m(t) \rangle$ is time dependent, e.g., when $m(t)$ follows a distribution $P(m) \propto m^{-\lambda}$. The behavior of $P(k)$ changes significantly as λ is varied: for $\lambda > 3$, the network has a scale-free distribution belonging to the BA class as predicted by the mean field theory; for smaller values of λ it shows different behavior. Characteristic features of the clustering coefficients in both models have also been discussed.

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I. INTRODUCTION

In many real-world networks which evolve in time, the number of links show a nonlinear growth in time [1]. Examples of such network are the Internet [2], World Wide Web (WWW) [3], collaboration [4], word web [5], citation [6,7], metabolism [8], gene regulatory network [9,10], etc. The number of links may increase in a twofold way: new nodes may tend to get attached to more nodes as the size of the network increases, and second there may be additional links generated between the older nodes in a nonlinear fashion as shown in Fig. 1. These two factors may be present either singly or simultaneously resulting in the accelerated growth. In some networks such as the citation and the gene regulatory network, new links between older nodes are forbidden and therefore only the first scheme is valid, while in collaboration network or Internet, the second factor is dominating.

The case when the new node gets a fixed number of links but older nodes get new links in a nonlinear fashion has been considered in both isotropic and directed models of growing networks [4,5,11], showing that it is distinct from networks with a linear growth rule. Here, the number of links generated between the older nodes is considered to have a power-law growth in time. This choice is made because the assumption that scale-free behavior (which is a desirable feature of networks) is present in a network with accelerated growth has been argued to be consistent with a power-law growth of links [11]. The evolution of the networks in these models of accelerated growth was made using a preferential attachment scheme as in the Barabási-Albert (BA) network [12]. In the directed network, this rule is modified by allowing an additional parameter, the initial attractiveness, in the attachment rule [13]. There is also an alternative way of achieving a scale-free network proposed by Huberman and Adamic [14] which has been used for modeling the Internet network with accelerated growth [15].

The preferential attachment scheme in the original BA network is the simple rule that the incoming nodes get attached to the ith node according to the probability

$$\Pi_i = \frac{k_i}{\Sigma k_i}, \quad (1)$$

where k_i is the degree (number of connections) of the ith node. This leads to the result that the number of links k is distributed according to

$$P(k) \propto k^{-\gamma} \quad (2)$$

for large k. Let the ith node be introduced at time t_i. Usually one node at a time is introduced such that $i=t_i$. The degree of the ith node at a later time t is then denoted by $k(t_i,t)$, which in the BA model can be estimated as

$$k(t_i,t) = \text{const} \times \left[\frac{t}{t_i}\right]^{\beta}. \quad (3)$$

In general, the exponent β describes the variation of $k(t_i,t)$ with t_i^{-1}, however in the case of the BA model, the behavior with t is also given by the same exponent. The values of the exponents can be obtained in the BA model exactly: $\gamma=3$ and $\beta=1/2$, satisfying the relation $\beta(\gamma-1)=1$. This relation holds good in a more generalized version of the BA model as well [13]. In the BA network, the incoming node gets a fixed number of links m_0 and the characteristics of the network do not depend on the specific value of m_0. No

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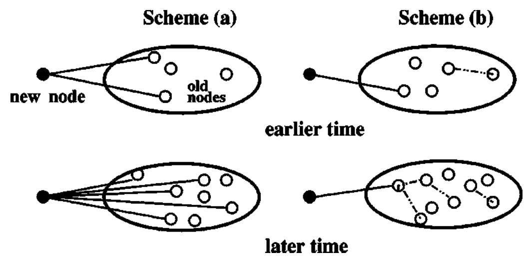

FIG. 1. The way accelerated growth takes place: in scheme (a), followed in this paper, the new node gets an increased number of links as time progresses, no new link between old nodes are allowed. In scheme (b), the new node gets a single link and old nodes get new links (shown by dashed lines) with the total number of links following a nonlinear growth in time. The most general case is a combination of the two schemes.

new link between older nodes is allowed. The models of accelerated growth considered so far assume that older nodes get new links [scheme (b) in Fig. 1], which is a sufficient departure from the original Barabási-Albert model. We consider the simpler case where the number of outgoing links $m(t)$ is no longer a constant but a function of time and no new link between older nodes is allowed [scheme (a) of Fig. 1]. This is also a realistic scheme as one can expect the number of attachments of a new node to increase when it is exposed to a larger environment. We have considered both deterministic and stochastic rules governing the form of $m(t)$.

The focus of the present paper is on the behavior of the various degree distributions in the networks with accelerated growth. A brief discussion of the clustering properties of the networks has also been made. The clustering coefficient $\mathcal{C}_i$ measures the number of links between the neighbors of the ith node. Here we have estimated the average clustering coefficient $\mathcal{C}(t)$ in a network evolved upto time t (equal to the number of nodes) and also the the average clustering coefficient $\mathcal{C}(k)$ of nodes with degree k. These quantities have been shown to have interesting properties in networks [16].

In Sec. II, the deterministic picture is discussed where the number of outgoing links increases in time in a deterministic manner and in Sec. III, stochastic models are considered in which the number of outgoing links is a random variable. In the last section, the results are summarized and discussed.

II. DETERMINISTIC MODEL

Let the number of nodes in a growing network be $N(t)$ at time t. In the deterministic model, we take the number of outgoing links available to an incoming node to be $m(t) = N(t)^\theta$. In a network which is grown by introducing one node at a time, $N(t)=t$, and therefore $m(t)=t^\theta$, ensuring an accelerated growth in the number of links in the network. The links are made according to Eq. (1) as usual.

One can obtain an expression for $k(t_i,t)$ and $P(k)$ using a continuum theory following Ref. [17]. Here the rate of change of $k(t_i,t)$ is taken proportional to $k_i/\Sigma k_i$. Going to the continuum limit the total number of links is $\int t^\theta d\theta = t^{1+\theta}/(1+\theta)$ at time t, and the equation governing $k(t_i,t)$ takes the form

$$\frac{\partial k(t_i,t)}{\partial t} = \frac{(\theta+1)k(t_i,t)}{2t}, \tag{4}$$

leading to

$$k(t_i,t) = t_i^\theta \left[\frac{t}{t_i}\right]^{(1+\theta)/2}. \tag{5}$$

In the last equation the boundary condition $k(t_i,t=t_i)=t_i^\theta$ has been inserted. From the above equation, we find that $\beta=(1-\theta)/2$. That $k(t_i,t)$ is not a function of t/t_i alone is a result significantly different from the BA network. The degree distribution at time t shows the following behavior:

$$P(k) = \frac{2}{1-\theta} k^{-1-2/(1-\theta)} f(t). \tag{6}$$

where $f(t)=[1/(1+t)]t^{(1+\theta)/(1-\theta)}$. The value of γ is therefore given by

$$\gamma = 1 + \frac{2}{1-\theta}. \tag{7}$$

In this model, there are two known limits: $\theta=0$ corresponds to the BA model and $\theta=1$ corresponds to a fully connected network (i.e., an N-clique). For the BA model, the degree distribution is a power-law distribution while for the fully connected network, $P(k)$ is a delta function. This implies the possibility of the existence of a "critical" θ_c where a peak occurs in the degree distribution for the first time. We conduct numerical simulations to explore this possibility.

In the simulation, nodes are added one by one. A specific number of links are assigned to each incoming node [$m(t) = t^\theta$: the nearest integer is chosen] and links are made by the preferential attachment scheme. For larger values of θ the network becomes highly connected and it takes a lot of time to generate it. Hence we restrict to values of $\theta \leqslant 0.7$ and to times $t \leqslant 4000$; the latter is also the size of the network. The results show complete agreement with the analytical results as far as the decay of the degree distribution for large k is concerned (Fig. 2). A growth of the distribution for small k values is noted as well. This growth seems to be faster than exponential as θ is made larger. This fast growth suggests that the form of $P(k)$ may be assumed to be $P(k) \sim k^{-\gamma}\Theta(k-k_c)$ where k_c is the value at which $P(k)$ is maximum. The normalization of $P(k)$ can then be done by making $\int_{k_c}^{\infty} P(k)dk = 1$. Substituting $P(k)$ from Eq. (6) and following the above normalization procedure, we get an estimate of k_c as a function of t, which is precisely $k_c = t^\theta$ for large t. The numerical results for discrete systems also agree with the above scaling, the agreement becoming better for larger values of θ.

Interestingly, we find that a peak is present in the degree distribution even for values of $\theta < 1$ which indicates the existence of θ_c. Analyzing the numerical results for small values of θ, we find that the peak in the distribution occurs as soon as θ is made nonzero. This is established by the fact that as the network size is made larger (i.e., the time to which

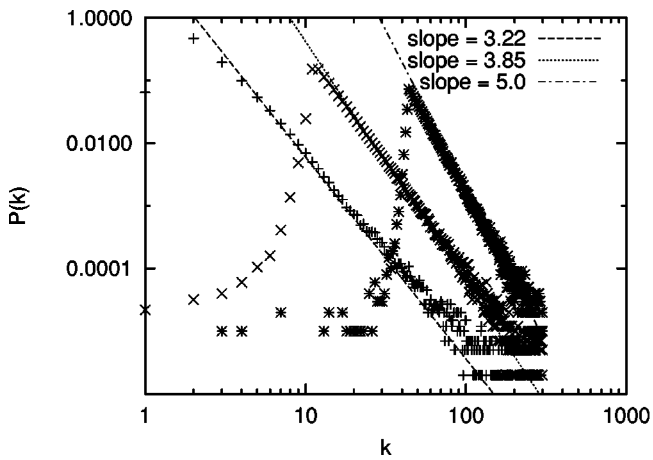

FIG. 2. The normalized degree distributions in the deterministic network for $\theta = 0.1, 0.3$, and 0.5 are shown along with straight lines in the log-log plot which have slopes according to Eq. (6). For $\theta = 0.1$ and 0.3, $t = 4000$ and for $\theta = 0.5$, $t = 2000$.

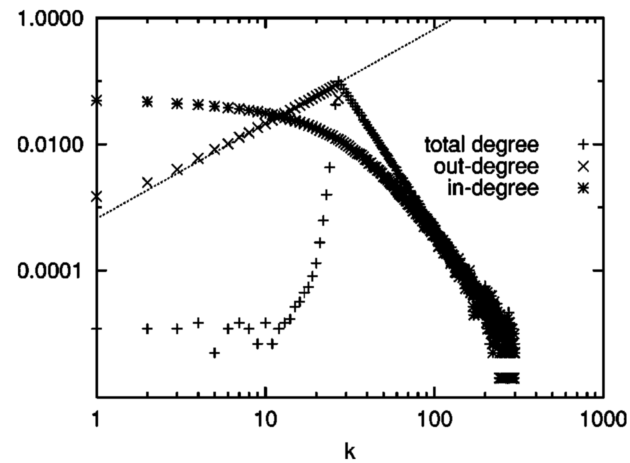

FIG. 3. The in-degree, out-degree, and total degree distributions for $\theta = 0.4$ are shown. $P_{out}(k_{out})$ is fitted with the calculated slope given by Eq. (8).

the network is evolved is made larger) the peak becomes sharper and $P(k)$ decreases for small k values. Hence we conclude that $\theta_c = 0$.

Although in this network $P(k)$ has been calculated from the total degree of the nodes, we notice an interesting behavior of the distributions of the in-degree k_{in} and out-degree k_{out} taken separately. It may be mentioned here that these distributions have also been obtained for real networks and in many of them it is seen that these are also scale-free with distinct exponents [18]. The number of outgoing links in the present network is deterministic and has the following feature: the first n_1 nodes have out-degree 1, the next n_2 have out-degree 2, and so on (this is due to the discrete nature of the network). For $\theta = 0$, all nodes have out-degree 1 and its distribution $P_{out}(k_{out})$ is a delta function (n_1 equals the number of nodes in the system and all $n_i = 0$ for $i \neq 1$). For $\theta = 1$, $n_1 = 1, n_2 = 1, n_3 = 1$, etc., and the out-degree distribution is a flat one. For intermediate values of θ, the exact form of $P_{out}(k_{out})$ can be easily found out. Let t be the first time when the out-degree of the incoming node is k_{out} and $t + \Delta t$ the time at which the out-degree increases to $k_{out} + 1$. Clearly $\Delta t = P_{out}(k_{out})$ and since $k_{out} = t^{\theta}$, we have $\theta t^{\theta - 1} \Delta t = 1$. Therefore

$$P_{out}(k_{out}) \propto \frac{1}{\theta} k_{out}^{(1-\theta)/\theta}. \tag{8}$$

Hence we find that the out-degree distribution actually grows with the degree, a result which is also verified in the numerical simulations (Fig. 3). Since for $\theta < 1$, the out-degree never assumes a very large value in a network of finite size, we also note that for large k, the contribution to $P(k)$ is mainly from k_{in}. Consequently, we expect that $P_{in}(k_{in})$ will have a power-law tail with γ given by Eq. (7). This is also confirmed numerically. The growing region of the total degree distribution for small k is accounted for by the growing out-degree distribution as in this region both k_{in} and k_{out} contribute.

The behavior of $k(t_i, t)$ is also in complete agreement with the theoretical results: plotting the scaled $k(t_i, t)/t^{(1+\theta)/2}$ against t_i for different values of t, a data collapse is obtained. This is shown in Fig. 4 for $\theta = 0.4$. The agreement with the continuum results becomes better as t_i increases.

All clustering coefficients at $\theta = 0$ are zero as no loops are allowed in this case. As θ is increased, $C(t)$ (with fixed t) shows an increase as expected. The increase is not very sharp at small values of θ, e.g., for θ as high as 0.5, $\mathcal{C}(t) = 0.038$ for $t = 2000$. Since $\mathcal{C}(t) = 1$ for $\theta = 1$, it is expected that $\mathcal{C}(t)$ will show a sharper increase for larger values of θ; it is however difficult to simulate networks in this range of values of θ to check this behavior. $\mathcal{C}(t)$ as a function of t shows a behavior similar to the BA model; it decreases with t (at least upto $\theta = 0.4$; we have not studied this variation beyond this value). This decrease is also expected to have a dependence on θ [in the limit $\theta = 1, \mathcal{C}(t)$, is independent of t]. We have not, however, attempted to study in detail the dependence of $\mathcal{C}(t)$ with t as θ is varied.

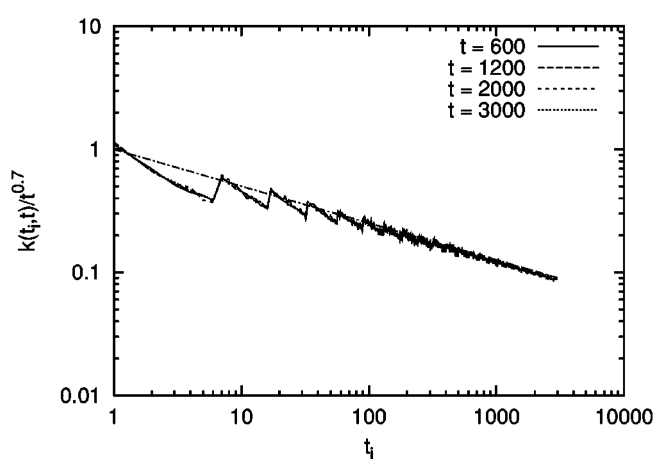

FIG. 4. The scaling plot for $k(t_i, t)$ is shown for $\theta = 0.4$. Here $k(t_i, t)$ has been scaled by $t^{0.7}$ as $k(t_i, t)$ varies as $t^{(\theta+1)/2}$ according to Eq. (5). The straight line is drawn with slope 0.3 as $\beta = (1-\theta)/2 = 0.3$ here.

We have also calculated $\mathcal{C}(k)$ when t is kept fixed which again shows a behavior similar to the BA model for large values of k, i.e., for nonzero values of θ,$\mathcal{C}(k)$ becomes a constant. This constant is a function of θ and we find that $\mathcal{C}(k) \sim \theta^2$ for large k gives a good fit to the data.

III. STOCHASTIC MODELS

The assumption that an incoming node gets attached to a fixed number of nodes at a given time in the deterministic model is somewhat artificial. In most social networks, the outgoing links also have a distribution which usually shows a decay [18]. Hence one should consider randomness in the number of outgoing links in a realistic manner such that the number of outgoing links m is not fixed at time t but is a stochastic variable. In fact, m can be a stochastic variable even in the conventional BA model by not putting any restriction on the number of outgoing links. This can be achieved by simply allowing each existing node the possibility to get attached to the incoming node according to the probability given in Eq. (1). However, it is known that making m random in this way does not change the BA universality class. This happens because even though m is random, $\langle m(t)\rangle$, the mean value is practically independent of time. Thus it is possible to replace m_0 by $\langle m(t)\rangle$ in the rate equation for $k(t_i,t)$ [17] and get the same results as in the BA model. Such a replacement is meaningful as long as the fluctuations are negligible which is true in the unrestricted BA case. This we have checked by numerical simulations also.

To get a stochastic model in which $\langle m(t)\rangle$ is a function of time, we let $m(t)$ follow a distribution which depends on the number of nodes $N(t)$ present at that time. The choice of the distribution can be done in many ways. However, we find that in many real networks, where the distribution of the out-degree has been done, the distribution shows either a power law (e.g., the WWW or phone-call network) or an exponential (e.g., as in the citation network) tail [18]. This study has been done in networks evolved for a sufficient duration of time; here we assume that the same kind of distribution is valid for intermediate times. The dependence on the number of nodes present in the system at time t occurs by putting the upper bound of $m(t)$ equal to $N(t)$.

Taking an exponentially decaying distribution of $m(t)$, however, again gives nothing new. This is because the mean value $\langle m(t)\rangle$ becomes time independent (within a short time) and therefore we get the BA network again. Thus we focus our attention on the model in which $m(t)$ follows a power-law distribution, i.e., $P_m(m(t)) \propto m(t)^{-\lambda}$ with $1 \leqslant m(t) \leqslant N(t)$. Again we take $N(t)=t$, i.e., one node is being added at a time.

In this model $\langle m(t)\rangle$ will have the following behavior:

$$\langle m(t)\rangle \sim t \quad (\lambda < 1) \tag{9a}$$

$$\sim t^{2-\lambda} \quad (1 < \lambda < 2) \tag{9b}$$

$$\sim O \quad (1)(\lambda > 2). \tag{9c}$$

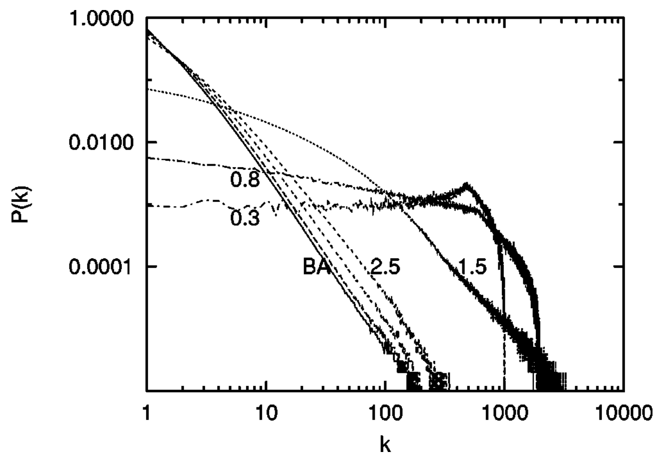

FIG. 5. The degree distributions for the stochastic model for several values of λ (denoted by the labels) are shown. In between the BA curve and the curve for λ=2.5, the two unlabeled curves correspond to λ=4.0 and λ=3.0. For higher values of λ, the network is grown to 4000 nodes while for the lowermost λ, we have maximum 1000 nodes.

Assuming that m can be replaced by its mean $\langle m(t)\rangle$, the continuum theory discussed in the last section can be used for the stochastic model as well once we define an effective θ from the above equations. Thus $\theta_{eff}=0$ for $\lambda > 2$, $\theta_{eff}=1$ for $\lambda < 1$, and for $1<\lambda<2$, $\theta_{eff}=2-\lambda$. We should not, however, expect this "continuum mean field theory" to be valid if the fluctuations are not negligible. An estimate of the fluctuation in m can also be made which shows that it increases as λ decreases and cannot be neglected for $\lambda<3$.

We use numerical simulations to find out the validity of the continuum mean field theory in the stochastic models. Plotting $P(k)$ vs k (Fig. 5) for several values of λ, we find that for large values of λ it is indeed BA-like. As λ is decreased it deviates from the power-law behavior. It is difficult to locate the exact value of λ where the change in behavior occurs, but clearly, the scale-free behavior observed for $\lambda>3$ is no longer valid for values of λ above 2 but less than 3. For $2<\lambda<3$, the fluctuations $(\langle m^2\rangle - \langle m\rangle^2)$ scale like $N(t)^{3-\lambda}$, which means that it becomes stronger for large $N(t)$ and for λ closer to 2. Increasing $N(t)$ to large values is difficult as it takes a long time to generate the network. So we analyze the behavior of $P(k)$ for λ close to 2, e.g., at $\lambda=2.2$ and 2.5, to check the role of fluctuations. For both these values, the behavior of $P(k)$ agrees better with a stretched exponential behavior: $P(k) \sim \exp(-ak^b)$ albeit with very small values of $b(b \sim 0.10)$. For $1<\lambda<2$, the deviation from power law is quite clear. As λ is made smaller than 2, the behavior of $P(k)$ is no longer monotonic and for small values of k it can be fitted to a stretched exponential function with b depending on λ. The range of k over which $P(k)$ follows a stretched exponential behavior shrinks beyond $\lambda=1.5$ where it shows a sharp drop (see for instance the curves for $\lambda=1.2$ and 1.0 in Fig. 6).

For values of $\lambda<1$, the fluctuations increase rapidly and the network takes very long time to be generated as it becomes more and more clustered. $P(k)$ shows a slow decay for $0.5<\lambda<1$ over a long range of k and drops sharply as k

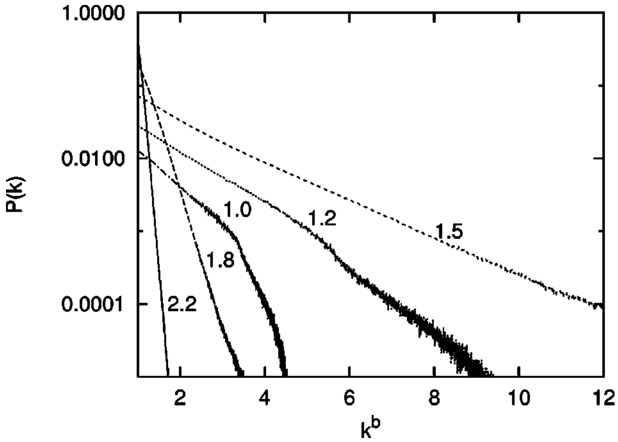

FIG. 6. The degree distribution is plotted against k^b to show the stretched exponential behavior for several values of λ (the labels denote the values of λ). The values of b are 0.1, 0.20, 0.42, 0.3, and 0.2 for $\lambda = 2.2, 1.8, 1.5, 1.2, 1.0$, respectively.

approaches the total number of nodes in the network. The distribution becomes flatter as λ is decreased. Below $\lambda = 0.5$, a weakly growing region emerges. In fact, a peak appears and becomes sharper as λ is made smaller. The position of the peak, as t is made larger, also shifts towards larger k, e.g., for $\lambda = 0.3$, the peak is at $k \sim 500$ for $t = 1000$ and at $k \sim 1000$ for $t = 2000$. A systematic study demands a very large amount of computer time and we have not attempted it. Nor do we try to fit the data to any familiar form because of the large amount of fluctuations.

As in case of the deterministic network, here too one can find out the individual distributions P_{out} and P_{in}. As expected, P_{out} shows a power-law decay with exponent λ. P_{in} almost coincides with the total degree distribution as the network becomes larger.

The behavior of $k(t_i, t)$ has also been studied in this network. Again we find a deviation from the power-law behavior as λ becomes less than 3. As λ is made smaller, $k(t_i, t)$ becomes flat as expected.

The clustering coefficients have been estimated in this model as well. $\mathcal{C}(t)$ as a function of λ shows a increase as λ is decreased as expected [e.g., for $\lambda = 3.0$, $\mathcal{C}(t) = 0.029$ and for $\lambda = 1.5$, $\mathcal{C}(t) = 0.679$ for $t = 1000$]. As a function of t, $\mathcal{C}(t)$ shows a decrease with t for $\lambda > 2.0$. However, as λ is made smaller, for the values of $t \leqslant 2000$ at least, $\mathcal{C}(t)$ shows an increase with t. We conjecture that $\mathcal{C}(t)$ becomes independent of t for large t values for $\lambda < 2.0$. Here $\mathcal{C}(k)$ shows a logarithmic decay with k: $\mathcal{C}(k) \sim a - b \ln(ck)$ with the values of a, b depending strongly on λ and $c \simeq 1$ for all λ. Both a and b approach zero for very large λ. As λ is decreased both a and b increase indicating a larger value of $\mathcal{C}(k)$ together with a sharper decay.

IV. SUMMARY, COMMENTS, AND DISCUSSIONS

We have considered both deterministic and stochastic models of growing networks with preferential attachment in which the number of incoming links is a function of time. In both models we have followed simple rules of evolution with a single tunable parameter. The main results obtained in the deterministic model are as follows:

(1) A power-law decay of the total degree distribution $P(k)$ for large k is obtained which is consistent with the continuum theory.

(2) A growing region in $P(k)$ for small k is also observed. This is due to the behavior of the out-degree distribution which has a power-law increase. A peak is obtained in $P(k)$ for any $\theta > 0$. The peak position k_c is found to vary as t^{θ}.

(3) The values of the exponents γ and β can be obtained. They satisfy the relation $\beta(\gamma - 1) = 1$ as in a general BA model.

(4) $k(t_i, t)$ can be estimated. It is not a function of t/t_i as in the BA model.

(5) The degree distribution $P(k)$ is also found to be nonstationary, i.e., dependent on the time upto which the network has been evolved.

The main results of the stochastic model are as follows:

(1) The behavior of $P(k)$ depends on the value of λ. The mean field theory predicts a transition at $\lambda = 2$ above which it becomes BA-like. The numerical simulations show that for $\lambda > 3$ the degree distribution is scale-free with $\gamma = 3$.

(2) The degree distribution loses its power-law decay nature at values of $\lambda < 3$ and assumes a stretched exponential form: $P(k) \propto \exp(-ax^b)$. However, the deviation from the scale-free behavior is marginal, as indicated by the low value of b in the region $2 < \lambda < 3$. Hence this may be a correction to the BA scaling and therefore, it may not be correct to conclude that a phase transition has occurred at $\lambda = 3$. Clearcut inconsistency with the mean field theory is observed for smaller values of λ when the stretched exponential behavior of $P(k)$ becomes more pronounced. For very small values of λ, a weakly growing region in $P(k)$ emerges. The data, with a lot of fluctuations, are difficult to handle in this region. However, this region is not of much interest to us as for real networks, such small λ values seem unrealistic.

(3) The power-law decay behavior of $k(t_i, t)$ observed for $\lambda > 3$ also becomes invalid as λ is made smaller.

(4) For large values of λ, the degree distribution is stationary while for small λ (presumably for $\lambda < 0.5$) it becomes dependent on the size of the network.

A straightforward comparison of the above two models shows that the stochastic model is closer to real-world networks. This is concluded from comparison with real-world data: the out-degree distributions in phone call, citation, etc., networks show a decay, either power law or exponential, while in the deterministic model, we find a growing behavior. Even though we get a power-law decay of $P(k)$ in the deterministic model, the exponent γ is always >3 while in most real networks, it is closer to 2 and can be even less than 2. The stochastic model on the other hand has an out-degree distribution which shows a power-law decay. It also shows that the scale-free behavior can vanish even in a growing network with preferential attachment. In fact the stretched exponential behavior of the degree distribution observed for small degrees is reminiscent of the behavior of the degree distribution in citation network [6]. The stochastic model is also of theoretical interest as it indicates the existence of transitions at finite values of λ. However, the deterministic model has its own merits: it is easy to construct and a number of results for it can be obtained analytically in the continuum limit. The peak obtained in the total degree distribution in this model may be compared to similar peaks obtained in some real-world networks, e.g., the coauthorship network [18] or the Indian railways network [19].

The citation network is perhaps the most appropriate network which the stochastic model emulates. However, the citation network is essentially a directed network. Therefore we have also considered a specific directed model in the stochastic case, which follows the attachment rule as in Ref. [13] and where $P_m(m)$ has a peak and an exponential decay [7]. The results show that it has a power-law decay in the in-degree distribution with $\gamma \sim 2$ (not much different from the case when there is no accelerated growth [13]). There are some available data on citation network [6,7] but the data and the analyses are not sufficiently general to compare with simulation data. In this context it may be mentioned that in a model to simulate the gene regulatory network, which is a directed network with accelerated growth, the outgoing links also show a power law with exponent close to 2 in good agreement with theory [10].

In conclusion, we have tried to construct models of accelerated growth where the number of outgoing links increase with the network size following certain prescriptions. A continuum theory can be formulated for the deterministic case which can be applied in the stochastic case in the mean field limit. Based on the different results obtained in the models we conclude that the stochastic model is closer to real-world networks.

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